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THE EFFECT OF SMALL DISSIPATION ON THE ONSET OF ONE-DIMENSIONAL SHOCK WAVES†

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It is demonstrated that the onset of shock waves of "general position" accompanying the one-dimensional motion of an isentropic gas of low viscosity is described by Il'in's special solution of the Burgers equation. It is shown that it describes the generation of shock wave in the one-dimensional Voigt and Khokhlov–Zabolotskii–Kuznetsov models. © 2001 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

An analysis of the spontaneous formation of shock waves, which Riemann concluded could possibly exist as a result of his investigation of the solutions of the equations for the one-dimensional motion of an isentropic gas ($h\alpha(h) \geqslant 0$ is the square of the velocity of sound in the gas)

$$h_T + (hv)_X = 0, \quad v_T + vv_X + \alpha(h)h_X = 0 \tag{1.1}$$

as before remains one of the urgent problems in fluid dynamics.

Within the framework of system (1.1), shock waves are described by discontinuous solutions. When account is taken of dissipation, narrow transitional domains, in which the velocity v and the density h change smoothly, arise instead of discontinuities. In particular, an analysis of such a transitional domain has also been carried out in [1–3] for the solutions of the equations of the one-dimensional motion of an isentropic gas of low viscosity

$$h_T + (hv)_X = 0, \quad v_T + vv_X + \alpha(h)h_X = \varepsilon^4 v_{XX}/h \quad (\varepsilon \ll 1) \tag{1.2}$$

However, the complete pattern of the development of shock waves for solutions (1.2) has not so far been obtained. There has been no analysis of the processes involved in the onset of these shock waves. The same also applies to a number of other models with small dissipation [3, 4].

Il'in's result [5, Chapter 6] on the asymptotic form, when $\varepsilon \to 0$, of the solutions of the model equation

$$v_T + g(v)v_X = \varepsilon^4 v_{XX}$$

stands as an isolated case against this background. Il'in showed that, for this equation, the onset of shock waves of "general position" in the dominant order with respect to ε is given by a special solution of Burgers equation (τ and ξ are extended variables)

$$\Gamma_\tau + \Gamma\Gamma_\xi = \Gamma_{\xi\xi} \tag{1.3}$$

which is the logarithmic derivative

$$\Gamma(\tau, \xi) = -2[\ln \Lambda(\tau, \xi)]'_\xi \tag{1.4}$$

of a modification of the well-known Pearcy integral

$$\Lambda(\tau, \xi) = \int_R \exp(-(4\lambda\xi - 2\tau\lambda^2 + \lambda^4)/8)d\lambda \tag{1.5}$$

In Section 3 of this paper, it is shown, in particular, that, in the case of system (1.2), the onset of shock waves of "general position" is described, as before, by this solution of Eq. (1.3) (this conclusion was announced previously in [6]). Here, it is shown that it also describes the onset of shock waves in

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the case of the one-dimensional Voigt model and for one of the spatially – one-dimensional versions of the Khokhlov–Zabolotskii–Kuznetsov model.

The results of the numerical modelling of this universal special solution of Eq. (1.3) are then described in Section 4. Note that the numerical computational procedure which is used here is based very much on the special role, in the case of Γ, of the line $\xi = 0$ forming the so-called Maxwellian stratum M [7, p. 306] of the cusp catastrophe which is described by the equation

$$\xi - \tau f(\tau, \xi) + f(\tau, \xi)^3 = 0 \tag{1.6}$$

(A root of Eq. (1.6) is the dominant term of the asymptotic form Γ when $\xi^2 + \tau^2 \to \infty$ everywhere outside the ray $\xi = 0, \tau > 0$ [5, Chapter 6].)

We will start by analysing the behaviour of the solutions of system (1.1) in the neighbourhood of their points of gradient catastrophes, which is necessary as a preliminary stage for the arguments presented in Section 3. However, this analysis is also of value in its own right.

2. LOCAL ANALYSIS IN THE NEIGHBOURHOODS OF POINTS OF GRADIENT CATASTROPHES

Solutions of system (1.1) with a value $I = h_T v_X - v_T h_X$ which is not identically equal to zero are considered. (When $I \equiv 0$, the velocity is a function of the density [8, p. 141].) In the case of such v and h, Eqs (1.1) are locally reduced to linear equations by means of a transformation of the hodograph: the consideration of T and X as coordinates and h and v as independent variables gives the system

$$X_h = vT_h - \alpha(h)T_v, \quad X_v = vT_v - hT_h \tag{2.1}$$

It is well known that the zeros of the Jacobian

$$J = X_h T_v - X_v T_h = h(T_h)^2 - \alpha(h)(T_v)^2 \tag{2.2}$$

of the smooth mapping $(h, v) \to (X, T)$ correspond to the points where the derivatives of the solutions of system (1.1) of "general position" become infinite. However, this fact, apparently, has been previously invoked to analyse the singularities of the solutions of system (1.1) in the neighbourhoods of their gradient catastrophes. Precisely taking account of this in the spirit of the approach used earlier in [9] in a similar situation enables us, below, to obtain the complete asymptotic expansions of v and h at these points.

More accurately speaking, the Taylor expansions at the point (h_*, v_*) of the solutions of the linear equation

$$hB_{hh} + 2B_h = \alpha(h)B_{vv} \tag{2.3}$$

to which system (2.1) reduces using the relations

$$T = B_v, \quad X = -B - hB_h + vB_v \tag{2.4}$$

will be used to analyse the singularities of the solutions of system (1.1) in the neighbourhood of a point of gradient catastrophe (T_*, X_*) (corresponding to the point (h_*, v_*) where the Jacobian J vanishes).

Remark 1. The coefficients

$$b_{ij}(v_*, h_*) = \left.\frac{\partial^{i+j} B(v, h)}{i!\,j!\,\partial v^i \partial h^j}\right|_{v=v_*,\, h=h_*}$$

of the Taylor series of all possible solutions of Eq. (2.3) which are smooth at the points $(v = v_*, h = h_*)$ (for which Jacobian (2.2) optionally vanishes at these points) are functions of v_* and h_*. Hence, in addition to the constraints on $b_{ij}(v_*, h_*)$ following from the validity of Eq. (2.3), because of the parameters v_* and h_* it is possible to impose two further constraints on them (and, in the analysis of the solutions of system (1.1) of "general position", no more than two) which are expressed using equalities. Two such constraints are actually used below in order for Jacobian (2.2) and the constant g_{02} in relation (2.17) to vanish.

Bearing in mind the remark about the possibility of imposing two (and no more than two) constraints in the form of equalities with respect to b_{ij} which do not follow from Eq. (2.3), we will henceforth assume without any loss of generality (the general case reduces to that being considered using trivial substitutions) that $X_* = T_* = v_* = 0$ and that the coefficient α (h) of system (1.1) is expanded at the point $h = h_*$ in a series of powers of $\rho = h - h_*$

$$\alpha(h) = 4 + \alpha_1\rho + \sum_{k=2}^{\infty} \alpha_k \rho^k \tag{2.5}$$

In this paper, the case when $h_* > 0$ is considered and a gas of "general position" for which $\alpha_1 + 12/h_* \neq 0$.

Substituting the Taylor series

$$B = \sum_{j,j=0}^{\infty} b_{ij} v^i \rho^j + \ldots \tag{2.6}$$

into Eq. (2.3) and taking account of relations (2.4) and (2.5), we obtain the following relations between its coefficients

$$\begin{gathered} b_{00} = -h_* b_{01}, \quad b_{10} = 0, \quad 4b_{20} = b_{01} + h_* b_{02} \\ 4b_{21} + \alpha_1 b_{20} = 3(b_{02} + h_* b_{03}), \quad 12b_{30} = b_{11} + h_* b_{12}, \ldots \end{gathered} \tag{2.7}$$

The fact that Jacobian (2.2) vanishes when $v = 0$, $\rho = 0$ means that b_{11} and b_{20} are expressed here in terms of one another using one of the two formulae

$$4b_{20} = \pm h_*^{1/2} b_{11} \tag{2.8}$$

It follows from (2.5)–(2.8) that transformation (2.4) decomposes into the series

$$\begin{aligned} T &= b_{11}(\rho \pm h_*^{1/2} v/2) + (b_{11} + h_* b_{12}) v^2/4 + 2b_{21} v\rho + b_{12}\rho^2 + \ldots \\ X &= Tv - b_{11}(h_* v \pm 2h_*^{1/2}\rho) - (b_{21} h_* \pm h_*^{1/2} b_{11}/4) v^2 - \\ &\quad -2(b_{11} + h_* b_{12}) v\rho - 3(b_{02} + h_* b_{03})\rho^2 + \ldots \end{aligned} \tag{2.9}$$

In the subsequent analysis, it is convenient to change to the variables

$$Y = X - 2h_*^{1/2} T, \quad Z = X + 2h_*^{1/2} T \tag{2.10}$$

and the unknowns

$$R = v + 2h_*^{-1/2}\rho, \quad L = v - 2h_*^{-1/2}\rho \tag{2.11}$$

which are identical in order of magnitude to the deviations of the Riemann invariants [10, p. 171]

$$r = v + \int_{h_*}^{h} \left(\frac{\alpha(z)}{z}\right)^{1/2} dz, \quad l = v - \int_{h_*}^{h} \left(\frac{\alpha(z)}{z}\right)^{1/2} dz, \tag{2.12}$$

from their common zero value at a point of gradient catastrophe

$$\begin{aligned} r &= R + \frac{1}{128}\left(\alpha_1 - \frac{4}{h_*}\right) h_*^{1/2} (R - L)^2 + O(|R|^3 + |L|^3) \\ l &= L - \frac{1}{128}\left(\alpha_1 - \frac{4}{h_*}\right) h_*^{1/2} (R - L)^2 + O(|R|^3 + |L|^3) \end{aligned} \tag{2.13}$$

At the point $T = 0, X = 0$, the lines $Y = 0$ and $Z = 0$ touch the characteristic C_+ along which the invariant r is constant and, respectively, the characteristic C_- along which the invariant l is constant.)

In the new notation, relations (2.9) appear as

$$Z-Y=h_*b_{11}(R-L\pm(R+L))+\frac{1}{4}h_*^{1/2}(b_{11}+h_*b_{12})(R+L)^2+$$
$$+\frac{1}{4}h_*^{3/2}b_{12}(R-L)^2+h_*b_{21}(R^2-L^2)+\ldots \tag{2.14}$$

$$Z+Y=\frac{1}{4}h_*^{-1/2}(Z-Y)(R+L)-h_*b_{11}(R+L\pm(R-L))-$$
$$-\frac{1}{2}(h_*b_{21}\pm\frac{1}{4}h_*^{1/2}b_{11})(R+L)^2-\frac{1}{2}h_*^{1/2}(b_{11}+h_*b_{12})(R^2-L^2)-$$
$$-\frac{3}{8}h_*(b_{02}+h_*b_{03})(R-L)^2+\ldots \tag{2.15}$$

We will now consider in detail the case of a minus sign. In the case of this sign, addition of equalities (2.14) and (2.15) gives the equation (a_{ij} are constants)

$$Z=F(R,L)=-2h_*b_{11}L-\frac{1}{16}h_*^{1/2}b_{11}\left(1-\frac{1}{4}\alpha_1h_*\right)R^2+$$
$$+\frac{1}{8}h_*^{1/2}b_{11}\left(1-\frac{1}{4}\alpha_1h_*\right)RL+a_{02}L^2+\sum_{i+j=3}^{\infty}a_{ij}R^iL^j$$

in which in the situation of a "general position" $b_{11}\neq 0$.

When $b_{11}=0$ and $\rho=0$, the relations

$$T=\frac{1}{4}h_*b_{12}\upsilon^2(1+o(1)),\quad X=-h_*b_{21}\upsilon^2(1+o(1))$$

should follow from (2.7)–(2.9) which, on account of the fact that the constants b_{12} and b_{21} are non-zero (see Remark 1) would mean that, when $\rho=0$ and in the case of sufficiently small X and T, the function υ is not defined on two mutually orthogonal curves. However, this is impossible in the situation being considered since υ and ρ, which have a first-order discontinuity on the shock wave front, are continuous on each side of it [10, p. 35].

Since $F_L(0,0)=-2h_*b_{11}\neq 0$, then L from the equation $Z=F(R,L)$ is expressed in terms of Z and R in the form of the series

$$L=-\frac{Z}{2h_*b_{11}}+\frac{1}{128}h_*^{1/2}\left(\alpha_1-\frac{4}{h_*}\right)\left(R^2+\frac{ZR}{h_*b_{11}}\right)+c_{20}Z^2+\sum_{i+j=3}^{\infty}c_{ij}Z^iR^j \tag{2.16}$$

where c_{ij} are certain constants which are uniquely defined in terms of b_{ij} and h_*.

Substitution of series (2.16) into equality (2.14) gives the equation in R

$$Y=\frac{1}{64}h_*^{1/2}\left(\alpha_1+\frac{12}{h_*}\right)ZR+g_{20}Z^2+g_{02}R^2+\sum_{i+j=3}^{\infty}g_{ij}Z^iR^j \tag{2.17}$$

where the constants g_{ij} on the right-hand side of this equation are also specified in terms of b_{ij} and h_*. When $T=0$, this equation can be briefly written in the form of the relation

$$X=g_{02}R^2(1+o(1))$$

from which it is rapidly concluded that $g_{02}=0$ (otherwise the solutions of Eqs (1.1) would only be defined on one side of the line $X=0$ in the case of small values of X and T). In this case, according to Remark 1 in a "general position" situation, $g_{03}\neq 0$ and, consequently, the transformation

$$R=C_0(Z)Z+R_0+\sum_{j=2}^{\infty}C_j(Z)R_0^j \tag{2.18}$$

the coefficients $C_j(Z) = \sum_{j=0}^{\infty} C_{ij}Z^i$ of which are uniquely defined [11, pp. 45, 46 and 52], reduces (2.17) to the cubic equation

$$\delta(Y, Z) + \sigma(Z)R_0 + kR_0^3 = 0 \tag{2.19}$$

where

$$\delta(Y, Z) = Y + \sum_{i+j=2}^{\infty} \delta_{ij} Y^i Z^j \tag{2.20}$$

$$\sigma(Z) = \frac{1}{64} h_*^{1/2}\left(\alpha_1 + \frac{12}{h_*}\right)Z + \sum_{j=1}^{\infty} \sigma_j Z^j$$

and the constant $k = -g_{03}$ is such that $(\alpha_1 + 12/h_*)k > 0$. (The last inequality corresponds to the fact that, up to the moment of the gradient catastrophe, r is a continuous function of the variables T and X.)

Equation (2.19) does not have uniquely determined roots for all δ and σ. According to relation (2.18), in the situation being considered the root of Eq. (2.19), which is single valued outside the line of discontinuity of the solutions of system (1.1) and loses the first-order discontinuity on this line, should be taken as $R_0(\delta, \sigma)$. In the case of a function $R_0(\delta, \sigma)$ which has been defined in this way, the point ($\delta = 0, \sigma = 0$) is a point of gradient catastrophe. At the origin of coordinates ($Y = 0, Z = 0$), the gradient catastrophe will therefore also hold in the case of the functions $R(Y, Z)$ and $L(Y, Z)$, the asymptotic forms of which, when $Y^2 + Z^2 \to 0$, are given by formulae (2.18)–(2.20) and (2.16) respectively.

Remark 2. It follows from the well-known properties of shock waves and from relation (2.18) that the functions R, L and R_0 have no other points of gradient catastrophe in a sufficiently small neighbourhood of the origin of coordinates. According to the first of relations (2.13), the gradient catastrophe at the point ($Y = 0, Z = 0$) will also hold for the Riemann invariant $r(Y, Z)$. However, this is no longer so in the case of the second Riemann invariant. In fact, by virtue of the second of relations (2.13) and the boundedness of the fractions $\sigma/(3kR_0^2(\sigma, \delta)+\sigma)$ and $\delta/(3kR_0^2(\sigma, \delta)+\sigma)$ in the case of small σ and δ (when $\sigma \neq 0$, it is simplest of all to verify their boundedness by changing to the self-similar variable $s = \delta|\sigma|^{-3/2}$ and making the substitution $R_0(\delta, \sigma) = |\sigma|^{-1/2} g(s)$), the first derivatives of l are bounded quantities when $\delta^2 + \sigma^2 \to 0$.

The case of the choice of the plus sign in (2.8), to which a gradient catastrophe of just one of the Riemann invariants, the invariant l, corresponds, is investigated in an entirely similar way. In view of the fact that the analysis of this case, when compared with the analysis of the case of a gradient catastrophe of the invariant r, does not contain any fundamentally new features, it is not presented here. For similar reasons, the case of a gradient catastrophe of the invariant l is also not considered in the following section.

3. THE EFFECT OF SMALL DISSIPATION

The effect of the right-hand side of system (1.2). In this section the effect on the processes involved in the onset of shock waves of the right-hand side of system (1.2), which is not taken into account on transferring from (1.2) to (1.1), is investigated using the example of the case of a gradient catastrophe of the invariant r.

To do this, we write system (1.2), using the substitutions (2.11), in the form of the equations

$$R_T - L_T + \frac{1}{2}(4h_*^{1/2} + R - L)(R_X + L_X) + \frac{1}{2}(R + L)(R_X - L_X) = 0$$

$$R_T + L_T + \frac{1}{2}(R + L)(R_X + L_X) +$$

$$+ \frac{1}{2}h_*^{1/2}\left(4 + \frac{\alpha_1 h_*^{1/2}}{4}(R - L) + \sum_{k=2}^{\infty} \alpha_k h_*^{k/2}\left(\frac{R - L}{4}\right)^k\right)(R_X - L_X) =$$

$$= \frac{\varepsilon^4}{h_*}\left(1 + \sum_{n=1}^{\infty} (-1)^n h_*^{-n/2}\left(\frac{R - L}{4}\right)^n\right)(R_{XX} + L_{XX})$$

Their addition and then subtraction reduces these equations to the system

$$R_T + 2h_*^{1/2}R_X + \frac{1}{16}(12+\alpha_1 h_*)R\left(1+\sum_{i+j=1}^{\infty} p_{ij}R^iL^j\right)R_X +$$
$$+\frac{1}{16}(4-\alpha_1 h_*)\left[L(1+\sum_{i+j=1}^{\infty} f_{ij}R^iL^j)R_X +\right.$$
$$\left.+R\left(1+\sum_{i+j=1}^{\infty} h_{ij}R^iL^j\right)L_X - L\left(1+\sum_{i+j=1}^{\infty}\gamma_{ij}R^iL^j\right)L_X\right] =$$
$$= \frac{\varepsilon^4}{2h_*}\left(1+\sum_{n=1}^{\infty}(-1)^n h_*^{-n/2}\left(\frac{R-L}{4}\right)^n\right)(R_{XX}+L_{XX}) \tag{3.1}$$

$$L_T - 2h_*^{1/2}L_X + \frac{1}{16}(12+\alpha_1 h_*)L\left(1+\sum_{i+j=1}^{\infty} q_{ij}R^iL^j\right)L_X +$$
$$+\frac{1}{16}(4-\alpha_1 h_*)\left[R(-1+\sum_{i+j=1}^{\infty} \delta_{ij}R^iL^j)R_X +\right.$$
$$\left.+L\left(1+\sum_{i+j=1}^{\infty} f_{ij}R^iL^j\right)R_X + R\left(1+\sum_{i+j=1}^{\infty}h_{ij}R^iL^j\right)L_X\right] =$$
$$= \frac{\varepsilon^4}{2h_*}\left(1+\sum_{n=1}^{\infty}(-1)^n h_*^{-n/2}\left(\frac{R-L}{4}\right)^n\right)(R_{XX}+L_{XX})$$

in which γ_{ij}, δ_{ij}, f_{ij}, h_{ij}, p_{ij} and q_{ij} are defined in terms of b_{ij} and h_*.

It follows from relations (2.11), (2.16) and (2.18) – (2.20) that the leading terms of the asymptotic forms of v and $h - h_*$ are proportional to R. Taking account of this remark, using reasoning which is standard in the method of matched asymptotic expansions [5], it can be shown that, for the correct description of the behaviour of the solutions of system (3.1) in a small neighbourhood of the point $(T = 0, X = 0)$, it is necessary to carry out the scale transformations.

$$Y = \varepsilon^3\zeta,\quad Z = \varepsilon^2\theta,\quad R = \varepsilon A,\quad L = \varepsilon^2 C \tag{3.2}$$

Their form is dictated by the requirements that all the terms in (2.19) should be balanced and that account should be taken of both the non-linearity and the viscosity in (3.1).

A direct check shows that, changing in (3.1) to the variables (2.10) and subsequent expansions of (3.2) in principal order with respect to ε give the following system

$$A_\theta + \frac{1}{64}h_*^{1/2}\left(\alpha_1 + \frac{12}{h_*}\right)AA_\zeta = \frac{1}{8}h_*^{-3/2}A_{\zeta\zeta}$$
$$C_\zeta = \frac{1}{64}h_*^{1/2}\left(\alpha_1 - \frac{4}{h_*}\right)AA_\zeta - \frac{1}{8}h_*^{-3/2}A_{\zeta\zeta} \tag{3.3}$$

which is supplemented by the matching condition with (2.16) and (2.18): when $\theta^2 + \zeta^2 \to \infty$, outside a certain ray, which is directed towards positive T (the tangent at the point $(T = 0, X = 0)$ to the line of discontinuity of the solutions of Eqs (1.1)), the solution of the first of Eqs (3.3) has a root of the cubic equation

$$\zeta - h_*^{1/2}(\alpha_1 + 12/h_*)\theta A_0/64 + kA_0^3 = 0$$

as the leading term of the asymptotic form $A_0(\theta, \zeta)$, while the leading term of the asymptotic form C is expressed in terms of the solution of this equation by means of the formula

$$C_0(\theta,\zeta) = -\theta/(2h_*b_{11}) + h_*^{1/2}(\alpha_1 - 4/h_*)A(\theta,\zeta)_0^2/128$$

The substitutions

$$\zeta = d\xi, \quad \theta = 8d^2h_*^{3/2}\tau, \quad A(\theta,\zeta) = 8[dh_*^2(\alpha_1 + 12/h_*)]^{-1}\Gamma(\tau,\xi)$$

where $d = (2^9 h_*^{-6}(\alpha_1 + 12/h_*)^{-3}k)^{1/4}$, reduce $A(\theta, \zeta)$ to the function (1.4), (1.5), and $\Gamma(\tau, \xi)$ satisfies both the Burgers equation (1.3), in which these substitutions convert the first of the equations of system (3.3), and the condition into which the above condition on the leading term of the asymptotic form $A(\theta, \zeta)$ transforms at the same time. The uniquely defined solution of cubic equation (1.6) is the leading term of the asymptotic form of the function (1.4), (1.5) when $\xi^2 + \tau^2 \to \infty$ everywhere outside the ray ($\xi = 0, \tau > 0$).

The universality of Il'in's special solution. The applicability of Il'in's special solution when describing the processes associated with the onset of shock waves is not confined to the generalized Burgers equation and the system of equations of motion of a one-dimensional isentropic gas. This solution is a special function, which like the Pearcy function in linear problems with small dispersion [12], must, as the authors assume, describe the neighbourhood of the point where the shock waves originate in the case of an extensive range of one-linear problems with small dissipation.

As one of the examples which confirm this prediction, we may mention the system of equations [3, p. 248]

$$u_T - v_X = 0, \quad v_T - uu_X + \varepsilon^4 u_{XX} = 0$$

which is equivalent to one of the spatial – one-dimensional cases of the Khokhlov–Zabolotskii–Kuznetsov equation [13]. The arguments, which show the correctness of the fact that the processes associated with the onset of shock waves are also described in main order by function (1.4), (1.5), do not in any way differ in essence from those presented above.

The same function in the main series with respect to ε also describes the onset of shock waves of "general position" in the one-dimensional Voigt model [2] which corresponds to the equation

$$u_{TT} - \varphi(u_X)_X + \varepsilon^4\psi(u_X)_{XT} = 0,$$

in which $\varphi'(u_X) > 0, \psi'(u_X) > 0$. (This equation, which describes the motion of a non-linear viscoelastic body, also arises when considering the limiting case of a crystal lattice when the interaction force between neighbouring particles not only depends on the distance between them but also on its rate of change [2, p. 7]. System (1.2), after changing to the Lagrangian system and, then, to mass coordinates, reduces to a special case of this equation [2, p. 5].)

Wave catastrophes and the symmetry of integrable equations. Function (1.4), (1.5) is a representative of a whole class of functions [9, 14–21] which, being special solutions of integrable non-linear partial differential equations, have an importance in non-linear problems which is similar to the importance of the special functions of wave catastrophes (SFWC) [22, p. 535, 23] in the case of linear problems. These non-linear analogues of SFWC arise in an extensive class of singularly-perturbed problems in the study of the effect of initially discarded terms on wave catastrophes, occurring in approximations, which arise as the result of ignoring these terms. (The typical singularities of smooth mappings described by catastrophe theory correspond to them [7, 11].)

A general hypothesis was formulated in [16, 17] concerning these non-linear analogues of SFWC which, at the present time, finds ever new confirmations. One of its consequences is the fact that, in the case of the integrability of the partial differential equations, which the non-linear analogues of SFWC satisfy, after their derivation (which is similar to the derivation of Eq. (1.3) from system (1.2)), these non-linear analogues of SFWC must simultaneously be solutions of the uniquely written ordinary differential equations in which the independent variables of the partial differential equations are the independent variables.

Moreover, from the results obtained previously [16, 18], the inference inevitably arises that, in this case, the non-linear analogues of SFWC will probably also be solutions of the ordinary differential equations in which the remaining control parameters [11] of the corresponding wave catastrophe serve as the independent variables.

In particular, the fact, which is used in Section 4, that the ordinary differential equation

$$\xi - \tau\Gamma + 4\Gamma_{\xi\xi} - 6\Gamma\Gamma_{\xi} + \Gamma^3 = 0 \tag{3.4}$$

is satisfied by Il'in's solution, the correctness of which can be directly shown from relations (1.3)–(1.5), could be derived as a corollary from this general position which also holds in the case of the analogues of SFWC, which do not have explicit representations but satisfy integrable partial differential equations that do not allow of linearization. (Such as, for example, the Korteweg – de Vries equation, the non-linear Schrödinger equation and the Kadomtsev – Petriashvili equation. Special solutions of these equations, which are non-linear analogues of SFWC have been considered earlier [9, 14–21].)

It has been established [17] that the corresponding ordinary differential equations are the steady-state parts of the symmetries of the integrable partial differential equations [24, p. 368] (rules have been formulated in [17] which enable one to derive these ordinary differential equations from the form of the known symmetries of the integrable partial differential equations and from the corresponding characteristic of the catastrophe theory list). In addition, with the exception of the case of the simplest fold catastrophe, the non-linear analogues of SFWC are not self-similar solutions of the partial differential equations that arise, since the ordinary differential equations, which they simultaneously satisfy, will not be the steady-state parts of the symmetries of a classical type : the corresponding symmetries are higher (or, as one sometimes says, generalized) [24, p. 368].

Note that, at least in the case of a cusp catastrophe, the above-mentioned universality of the non-linear analogues of SFWC holds in the case of the solutions of integrable equations. (Together with Il'in's special solution of Burgers equation, universality holds, for example, in the case of its "dispersion" analogue, the well-known Gurevich – Pitayevskii special solution of the Korteweg – de Vries equation [19]. The special solutions of the non-linear Schrödinger equation, which are analogues of the Pearcy integral, provide two further examples [9, 15].)

The universality of these non-linear special functions reveals the importance of the relatively little known role of integrable partial differential equations which they play in problems of mathematical physics which are described using non-integrable equations with small parameters at the higher derivatives. The higher symmetries, which are only characteristic of integrable equations [24, p. 368], manifest themselves in a striking was in this case.

A version of the symmetry selection rules. Equation (3.4) was written in explicit form for the first time [20] as an illustration of a version which had been proposed (an alternative to that formulated earlier in [17]) of the symmetry selection rules of integrable equations, the exact solutions of the steady-state parts of which are the corresponding analogues of SFWC. These rules are:

(1) in the steady-state part of chosen symmetry and, also, in the partial differential equations themselves, it is necessary (sometimes, as in the case of the example in [9] after preliminary transformations) to take the "dissipation-free" ("dispersion-free") limit by discarding the higher derivatives;

(2) the leading term of the asymptotic form of the non-linear function being investigated must satisfy the equations which have been obtained.

For example, it follows from the form of the known symmetries of Eq. (1.3) [24, p. 380] and the requirement that the root of the cusp equation (1.6) $f(\tau, \xi)$ should be an exact solution of the "dissipation-free" limit of the steady-state part of the symmetry of Eq. (1.3) that this steady-state part is necessarily the sum of the steady-state part of the classical Galilean symmetry

$$1 - \tau\Gamma_{\xi} = 0$$

and the steady-state part of the third-order symmetry

$$4\Gamma_{\xi\xi\xi} - 6\Gamma\Gamma_{\xi\xi} - 6\Gamma_{\xi}\Gamma_{\xi} + 3\Gamma^2\Gamma_{\xi} = 0$$

which is independent of ξ and τ. The resulting ordinary differential equation is Eq. (3.4), differentiated with respect to ξ.

A comparison of what has been discussed here and the initial [17] versions of the symmetry selection rules shows that both have their own advantages and disadvantages. The advisability of using one or other of them depends on the specific details of the problems which arise.

4. SIMULATION OF THE BEHAVIOUR OF IL'IN'S SPECIAL SOLUTION

Ordinary differential equation (3.4) and the system of differential equation in the variable τ

$$4\Gamma_\tau = 2\Gamma\Gamma_\xi - \xi + \tau\Gamma - \Gamma^3$$
$$\Gamma_{\xi\tau} = 2\Gamma_\xi^2 - 1 + \tau\Gamma_\xi - \tfrac{1}{2}(\xi\Gamma - \tau\Gamma^2 + \Gamma^4) \quad (4.1)$$

on Γ and Γ_ξ, which is a corollary of the fact that Eqs (1.3) and (3.4) simultaneously hold, prove convenient to use for the simulation of the behaviour of Il'in's special solution.

The usual Runge–Kutta (RK) scheme was applied to system (3.4), (4.1). In this case, it is true that it was necessary to take account of the nuances associated with the stability of the problem of recovering the solutions using their power asymptotic form at one of the infinities. Let us say that the use of the RK method with initial data determined from the asymptotic form of Γ outside the line $\xi = 0$ leads to the fact that the condition $\Gamma(0, \tau) = 0$ is not satisfied.

Precisely because of this, the simulation of Γ was initially carried out in the Maxwellian stratum $M = (\tau \in (-\infty, \infty), \xi = 0)$. By virtue of the oddness of Γ with respect to ξ, system (4.1) in M is substantially simplified and reduces to the first-order ordinary differential equation

$$4\Gamma_{1\tau} = 2\Gamma_1^2 - 1 + \tau\Gamma_1 \quad (\Gamma_1(\tau) = \Gamma_\xi(\tau, 0)) \quad (4.2)$$

The initial condition in the numerical solution of Eq. (4.2) by the RK method was determined in this case from the correctness in the case of the function Γ_1 of the following asymptotic form when $\tau \to -\infty$

$$\Gamma_1 \approx \tau^{-1} - 6\tau^{-3} + 96\tau^{-5} + \ldots \quad (4.3)$$

The form of the leading term and the power character of the series on the right-hand side of (4.3) follows from formula (VI.4.24) in [5], and the actual values of the constants at the different powers of τ are most simply calculated by direct substitution of this series into Eq. (4.2).

Now, after the numerical solution of problem (4.2), (4.3) using the RK method, the Cauchy problem was solved, which is defined by ordinary differential equation (3.5) and the boundary conditions

$$\Gamma|_{\xi=0} = 0, \quad \Gamma_\xi|_{\xi=0} = \Gamma_1(\tau)$$

In the numerical calculation, the results of which are partially represented in Fig. 1 in the form of the dependence of Γ on ξ at different instants of time τ (only the domain $\xi > 0$ is shown since $\Gamma(\tau, \xi) = -\Gamma(\tau, -\xi)$, the asymptotic form (4.3) were replaced by the initial data

$$\Gamma_1|_{\tau=\tau_*} = \tau_*^{-1} - 6\tau_*^{-3} + 96\tau_*^{-5}$$

When $\tau_* \leqslant -15$, the results of the calculation were independent of the choice of the actual values of τ_* and gave the same pattern. The calculation is also stable with respect to the number of terms of the asymptotic form (4.3) taken into account and with respect to the length of the mesh step. Note that the monotonicity in the decrease in Il'in's special solution with respect to ξ, which is observed in the figure, corresponds to the monotonicity in the behaviour of the shock waves which have already been formed [2, pp. 204–207].

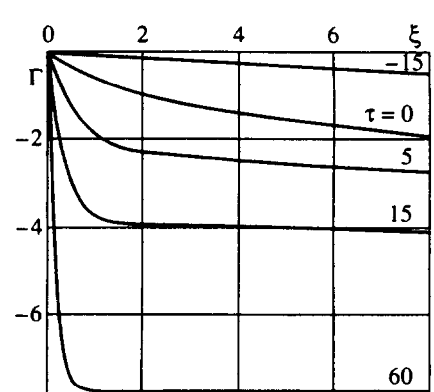

Fig. 1

Remark 3. The basic need when approximating the behaviour of the derivative of Il'in's special solution when $\xi = 0$ to use a "bottom-up" calculation (which corresponds to an increase in τ from $-\infty$ to ∞) and not to work in the opposite direction is associated with the instability of the recovery of the solution of Eq. (4.2) using a power asymptotic form when $\tau \to \infty$

$$\Gamma_1 \approx -\tau/2 + \ldots, \tag{4.4}$$

the correctness of which in the case of function (1.4), (1.5) follows from formula (VI.6.5) in [5]. In fact, together with the derivative of Il'in's special solution $\Gamma_\xi\ \tau, \xi = 0$, Eq. (4.2) also possesses a solution in the form of a perturbation theory series in powers of a small parameter μ

$$\Gamma_\xi(\tau, \xi = 0) + \mu R_1(\tau) + \mu^2 R_2(\tau) + \ldots$$

all of the terms of which are found from the solutions of a recurrent sequence of first-order ordinary differential equations. According to relations (4.3) and (4.4), all the non-trivial solutions of the first equation from this sequence

$$4R_{1\tau} = (4\Gamma_\xi(\tau, \xi = 0) - \tau)R_1$$

decay exponentially when $\tau \to \infty$ and increase exponentially when $\tau \to \infty$. This means that, in a "top-down" calculation in which the solutions of Eq. (4.2) with initial conditions determined from asymptotic form (4.4) are approximated, instead of the behaviour of the derivative of Il'in's special solution, the behaviour of the solutions of Eq. (4.2) is modelled, which strongly differ from this solution. An "bottom-up" calculation removes this deficiency.

Remark 4. The simplification in the Maxwellian stratum of the description of the cusp catastrophe special function is not only possible in the case of Il'in's special solution. For example, it is necessary to refine the comments [12, p. 176] on the fact that the Pearcy function

$$\int_R \exp(-2i(\xi p + 2\tau p^2 + \beta p^4))dp$$

is not expressed in terms of classical special functions: in the Maxwellian stratum $\xi = 0$, this SFWC is obviously expressed in terms of Weber–Hermite functions. This property of a Percy function is "inherited" by its generalization [15]. It has been noted [21], that the fourth-order ordinary differential equation in τ, which this generalized Pearcy function together with the non-linear Schrödinger equation

$$-iq_\tau = q_{\xi\xi} + 2\delta|q|^2 q$$

satisfy, reduces in the Maxwellian stratum $\xi = 0$ to the ordinary differential equation of Painlevé IV which is the non-linear analogue of the Weber–Hermite equation. The possibility of a simplified description of some SFWC and their analogues in Maxwellian strata is an additional manifestation of the separate character of the points of these strata which was observed in [7, 11] when investigating various wave catastrophes. It may prove useful when investigating the asymptotic forms of the non-linear analogues of SFWC and (an in this paper) in the simulation of their behaviour.

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